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PROJECTIVE STRUCTURES WITH FUCHSIAN HOLONOMY

WILLIAM M. GOLDMAN

A projective structure on a manifold S is a distinguished system of local coordinates modelled on a fixed projective space $\mathbf{P}$ in such a way that the local coordinate changes are locally projective. In this paper we will be mainly concerned with projective structures on manifolds of real dimension 2; thus $\mathbf{P}$ is either the complex projective line $\mathbf{CP}^1$ or the real projective plane $\mathbf{RP}^2$. If S is a topological surface then we shall speak of $\mathbf{CP}^1$-structures (resp. $\mathbf{RP}^2$-structures) on S; a manifold with a $\mathbf{CP}^1$-structure (resp. a $\mathbf{RP}^2$-structure) will be called a $\mathbf{CP}^1$-manifold (resp. an $\mathbf{RP}^2$-manifold).

It is well known that if M is a manifold with a projective structure modelled on a projective space $\mathbf{P}$, then there exists a pair (dev, φ) (unique up to projective automorphisms of $\mathbf{P}$), where dev: $\tilde{M} \to \mathbf{P}$ is a projective immersion, and φ is a homomorphism of $\pi = \pi_1(M)$ into the group of projective automorphisms of $\mathbf{P}$. The so-called developing map dev globalizes the coordinate charts defining the projective structure, while the holonomy homomorphism φ globalizes the coordinate changes. It is the purpose of this paper to classify projective structures on closed surfaces whose holonomy homomorphism is a fixed Fuchsian representation.

A *Fuchsian representation* of a discrete group π on $\mathbf{CP}^1$ is a faithful representation of π onto a discrete subgroup of $\mathrm{PSL}(2, \mathbf{C})$ preserving a disc Ω in $\mathbf{CP}^1$. Let φ be a Fuchsian representation φ of π on Ω. Using the Poincaré model for hyperbolic geometry, $\Omega/\varphi(\pi)$ has a natural hyperbolic structure, which we call the Fuchsian $\mathbf{CP}^1$-structure with holonomy φ. Conversely, every hyperbolic structure determines a Fuchsian $\mathbf{CP}^1$-structure in this way.

Similarly, suppose that φ is a representation of π in the group of projective transformations of $\mathbf{RP}^2$ (which we identify with $\mathrm{SL}(3, \mathbf{R})$). Then we say that φ is *Fuchsian* if φ is a faithful representation of π onto a discrete subgroup of

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$\mathrm{SL}(3, \mathbf{R})$ preserving a convex domain Ω bounded by a conic in $\mathbf{RP}^2$. Using the Klein-Beltrami model for hyperbolic geometry, we see that the $\Omega/\varphi(\pi)$ has a natural hyperbolic structure, called a Fuchsian structure; conversely a hyperbolic structure determines a Fuchsian $\mathbf{RP}^2$-structure in this way.

Using a construction due originally to Maskit [19], Hejhal [11], and Sullivan-Thurston [22] (all independently), one can modify Fuchsian structure by inserting annuli into Fuchsian structures split along simple closed curves. This surgery process, which we call "grafting," is described in detail in §1. We denote the set of all isotopy classes of disjoint collections of homotopically nontrivial simple closed curves on S by $\mathscr{S}$. Our main result on $\mathbf{CP}^1$-structures is the following:

Theorem C. *Let S be a closed surface and $\varphi: \pi \to \mathrm{PSL}(2, \mathbf{R}) \subset \mathrm{PSL}(2, \mathbf{C})$ a Fuchsian representation. Let M be a $\mathbf{CP}^1$-structure on S with holonomy φ, and let M_0 be the Fuchsian $\mathbf{CP}^1$-structure on (S) with holonomy φ. Then there exists a unique $\sigma \in \mathscr{S}$ such that M is obtained from M_0 by grafting along σ.*

Thus we may identify the set of developing maps (i.e., projective structures) with fixed holonomy φ with the discrete set $\mathscr{S}$.

Thurston [24] (see [2]) has shown that the deformation space $\mathbf{CP}^1(S)$ of all $\mathbf{CP}^1$-structures on S admits a natural description as a product $\mathscr{T}_S \times \mathscr{M}\mathscr{L}_S$, where $\mathscr{T}_S$ is the Teichmüller space of S and $\mathscr{M}\mathscr{L}_S$ is the space of all measured geodesic laminations on S. The space $\mathscr{M}\mathscr{L}_S$ has a natural structure as a piecewise linear manifold with integral coordinate changes, and the set $\mathscr{S}$ may be interpreted as the set $\mathscr{M}\mathscr{L}_S(\mathbf{Z})$ of integral points on $\mathscr{M}\mathscr{L}_S$. Thus under Thurston's description, the $\mathbf{CP}^1$-structures with Fuchsian holonomy may be identified with the set of integral points in $\mathscr{T}_S \times \mathscr{M}\mathscr{L}_S$. In particular this set is a countably infinite disjoint union of open $(6g-6)$-cells, where g is the genus of S.

The analogue of Theorem C for $\mathbf{RP}^2$-structures is more complicated. The first examples of exotic $\mathbf{RP}^2$-structures are due independently to Thurston [22] and Smillie [21]. Using Thurston's construction and the work of Nagano-Yagi [20] on affine structures, the classification of $\mathbf{RP}^2$-structures on the 2-torus was given in [6]. The developing maps of certain $\mathbf{RP}^2$-structures on the torus are described by equivalence classes of certain words in two symbols. A *positive word* in two symbols A, B is an element $w(A, B)$ of the free semigroup generated by A and B. We shall say that $w(A, B)$ is *completely even* if its exponent sums in A and B are each even. Let S be a closed surface. A *grafting data* on S, by definition, consists of an isotopy class of a disjoint collection Γ of homotopically nontrivial simple closed curves, no two of which are isotopic, together with a completely even positive word $w_\gamma(A, B)$ for each $\gamma \in \Gamma$. Let $\mathscr{G}\mathscr{D}$ denote the set of all such objects. In §§3 and 4, it is shown how to associate to a hyperbolic surface S and grafting data $\sigma \in \mathcal{GD}$ a new "grafted" $\mathbf{RP}^2$-structure on S. Our main result on $\mathbf{RP}^2$-structures is the following.

Theorem R. *Let S be a closed surface and $\varphi: \mathrm{SO}(2,1) \subset \mathrm{SL}(3,\mathbf{R})$ a Fuchsian representation. Let M be a $\mathbf{RP}^2$-structure on S with holonomy φ, and let M_0 be the Fuchsian $\mathbf{CP}^1$-structure on S with holonomy φ. Then there exists a unique $\sigma \in \mathcal{GD}$ such that M is obtained from M_0 by grafting along σ.*

Theorems C and R establish a bijection between developing maps of projective structures with Fuchsian holonomy and certain sets of simple closed curves weighted by elements of semigroups. For $\mathbf{CP}^1$-structures this semigroup consists of the positive integers, and for $\mathbf{RP}^2$-structures this semigroup consists of completely even positive words in two symbols. Thus we classify geometric structures in terms of intrinsic topological data on S.

As noted by Hejhal [11] and Sullivan-Thurston [22], the developing maps of grafted $\mathbf{CP}^1$-structures fail to be covering maps onto their images. Thus one consequence of our work is the construction of $\mathbf{RP}^2$-structures on all non-simply connected surfaces with pathological developing maps. As noted by Benzecri [1] and Sullivan-Thurston [22], $\mathbf{RP}^2$-structures on a manifold M yield affine structures on products of M with a circle. This enabled Sullivan-Thurston [22] and Smillie [21] to construct the first examples of affine structures on compact manifolds with pathological developing maps. Using these techniques we obtain the following:

Corollary. *Let M^3 be a closed 3-manifold homeomorphic to a product of a surface of genus greater than one with a circle. Then M^3 admits an affine structure whose developing map is a surjection onto the complement of the origin in $\mathbf{R}^3$, and is not a covering map. Furthermore examples exist where the affine structure is a dense subgroup of either $\mathrm{SO}(2,1) \times \mathbf{R}^+$ or $\mathrm{GL}(3,\mathbf{R})$.*

(For more information on affine structures, the reader is referred to Fried-Goldman-Hirsch [4] and Goldman-Hirsch [10].)

Finally we mention one last set of related examples. A *flat conformal structure* is a geometric structure modelled on the n-sphere and its group of conformal automorphisms. Kulkarni [17] showed that there is a natural operation of connected sums on such structures. Using this operation it is not difficult to construct flat conformal structures whose developing maps are surjective but not covering maps onto their images. Kuiper asked whether such examples could exist on 3-manifolds which are not connected sums. Using the work of Thurston on hyperbolic 3-manifolds, we show:

Proposition. *Let M^3 be a closed atoroidal 3-manifold with an incompressible surface which is not the fiber of a fibration of M over the circle. Then M admits a flat conformal structure whose developing map is surjective but not covering.*

In higher dimensions similar techniques yield the following.

Proposition. *Let M be a closed hyperbolic manifold which admits a closed totally geodesic hypersurface. Then M admits a flat conformal structure whose developing map is surjective but not covering.*

This paper is organized as follows. In §1 the Maskit-Hejhal grafting procedure is described. In §2 Theorem C is proved. The reader interested only in $\mathbf{CP}^1$-structures need read no further. In §3, the classification of $\mathbf{RP}^2$-structures on tori and annuli with real-split holonomy is described (following [6]). In §4 grafting of $\mathbf{RP}^2$-structures is defined and Theorem R is proved. In that section there is also a description of $\mathbf{RP}^2$-structures with holonomy in SO(2, 1) and various new examples of such structures are given. In §5, it is shown how grafting techniques can be used to construct flat conformal structures on hyperbolic manifolds with complicated developing maps.

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1. Grafting complex projective structures

1.1. Before describing the constructions, we establish the following notations. G will denote the projective group $\mathrm{PSL}(2, \mathbf{C})$, M will denote a $\mathbf{CP}^1$-manifold, $\tilde{M}$ will denote a fixed universal covering space of M, and π will denote the group of deck transformations of $\tilde{M}$. Fix a developing map $\mathrm{dev}: \tilde{M} \to \mathbf{CP}^1$. We shall denote the corresponding holonomy homomorphism by $\varphi: \pi \to G$.

The Maskit-Hejhal-Sullivan-Thurston construction. The basic construction we shall describe is due to Maskit [19], Hejhal [11], and Sullivan-Thurston [22] although our present treatment differs somehat from these references. Let $A \in G$ be a hyperbolic (including loxodromic) transformation. That is, the stationary set $\mathrm{Fix}(A)$ of A in $\mathbf{CP}^1$ consists of two points, one of which is an attractive fixed point for A, the other is a repelling fixed point for A. The cyclic group $\langle A \rangle = \{A^n : n \in \mathbf{Z}\}$ generated by A acts properly discontinuously on the complement $\Omega_A = \mathbf{CP}^1 \setminus \mathrm{Fix}(A)$. There exists a circle C in $\mathbf{CP}^1$ such that C and $A(C)$ bound disjoint discs each containing a point of $\mathrm{Fix}(A)$; the annular region bounded by two such circles is a fundamental domain for $\langle A \rangle$ on Ω_A. The quotient $T_A = \Omega_A / \langle A \rangle$ is a $\mathbf{CP}^1$-manifold diffeomorphic to a 2-torus. Such a $\mathbf{CP}^1$-manifold will be called a *Hopf torus*. The holonomy homomorphism $\pi_1(T_A) \cong \mathbf{Z} \oplus \mathbf{Z} \to \langle A \rangle \hookrightarrow G$ has cyclic kernel, generated by a simple closed curve which possesses a lift to the universal cover $\tilde{M}$ which develops to one of the boundary components of a fundamental domain. A simple closed curve in M which is not in this kernel has a component of its preimage which is an arc $\tilde{\gamma}$ in the universal cover which develops to an open arc in $\mathbf{CP}^1$ having endpoints the two fixed points $\mathrm{Fix}(A)$.

For each integer $n > 0$ let $\Omega_A^{(n)}$ denote the unique connected n-fold covering space of Ω_A. The projective automorphism A of Ω_A lifts to a unique projective automorphism $A^{(n)}$ of $\Omega_A^{(n)}$, such that the covering projection $\Omega_A^{(n)} \to \Omega_A$ is $\langle A \rangle$-equivariant. Hence the quotient $T_A^{(n)} = \Omega_A^{(n)}/A^{(n)}$ is another $\mathbf{CP}^1$-manifold with holonomy homomorphism isomorphic to that of T_A, but with a distinct developing map from T_A. These are the simplest examples of distinct $\mathbf{CP}^1$-structures on compact manifolds with the same holonomy.

1.2. Next consider a topological surface S and a homeomorphism $S \to M$ where M is a $\mathbf{CP}^1$-manifold. Let $\mathrm{dev}: \tilde{M} \to \mathbf{CP}^1$ be the developing map.

Definition. Let M be a $\mathbf{CP}^1$-manifold. A simple closed curve σ in M is said to be *admissible* on M if and only if

(i) The holonomy $\varphi(\sigma)$ around σ is hyperbolic (including loxodromic).

(ii) There exists a component of the preimage of $\tilde{\sigma}$ in the universal cover which develops to a simple arc in $\mathbf{CP}^1$ with endpoints the two elements of $\mathrm{Fix}(\varphi(\sigma))$.

Let σ be a simple closed curve in M whose holonomy is hyperbolic. Suppose σ is an admissible simple closed curve. Then $\mathrm{dev}(\tilde{\sigma})$ is a closed A-invariant subset of $\mathbf{CP}^1 \setminus \mathrm{Fix}(A)$ and the quotient $\sigma' = \tilde{\sigma}/\langle A \rangle$ is a simple closed curve inside the Hopf torus T_A. In particular σ has a tubular neighborhood $\mathrm{Nbd}_M(\sigma)$ in M which is projectively isomorphic to a tubular neighborhood of the corresponding curve σ' in the Hopf manifold $T_{\varphi(\sigma)}$ (whose holonomy generates the holonomy group of $T_{\varphi(\sigma)}$). This is exactly the situation needed for grafting.

Suppose σ is an admissible simple closed curve on M. Let $M|\sigma$ denote the surface obtained by splitting M along σ. The boundary of $M|\sigma$ consists of two simple closed curves σ_+, σ_- which are identified under a quotient map $\tilde{q}: M|\sigma \to M$ which is 1-1 on $\mathrm{int}(\tilde{M})$ and 2-1 on $\partial\tilde{M} = M_+ \cup M_-$. Our surgery procedure will begin with $\mathbf{CP}^1$-manifolds M, N and curves $A \subset M$, $B \subset N$ with a projective isomorphism between a tubular neighborhood of A in M and B in N. The union of $M|A$ and $N|B$ along the resulting identification of boundary components is a $\mathbf{CP}^1$-manifold which we shall call the *sum of A and B* (with respect to the isomorphism $\mathrm{Nbd}_M(A) \to \mathrm{Nbd}_N(B)$).

Let T_A be the Hopf torus with holonomy A and $T_A^{(n)}$ its n-fold covering. Let σ be a curve in T_A which has holonomy $A \in G$. We shall form the union of n T_A's along A. First $T_A|\sigma$ is a compact $\mathbf{CP}^1$-manifold with boundary homeomorphic to an annulus, whose holonomy homomorphism $\pi_1(T_A\,|\,\sigma) \cong \mathbf{Z} \to \langle A \rangle$ is an isomorphism. The universal covering space of $T_A|\sigma$ is projectively isomorphic to the universal covering space of the complement $\mathbf{CP}^1 \setminus \mathrm{Fix}(A)$. To form the n-fold sum consider the disjoint union of n copies of $T_A|\sigma$. Next identify boundary components to obtain the n-fold sum $T_A + \cdots + T_A$. Observe that the n-fold sum is projectively isomorphic to the n-fold covering space $T_A^{(n)}$. In particular by summing coverings of Hopf tori we obtain sum relations $T_A^{(m)} + T_A^{(n)} = T_A^{(m+n)}$, etc.

This identifies the set of $\mathbf{CP}^1$-structures with cyclic holonomy generated by A with the *semigroup of positive integers*. By summing these structures on annuli to structures on surfaces of higher genus we obtain new $\mathbf{CP}^1$-structures possessing holonomy isomorphic to that of the original structure.

1.3. Suppose M is a $\mathbf{CP}^1$-manifold with fixed developing map $\mathrm{dev}: \tilde{M} \to \mathbf{CP}^1$ and holonomy homomorphism $\varphi_M: \pi_1(M) \to G$ and suppose that σ is a curve in M whose holonomy $\varphi_M(\sigma) = A$ is hyperbolic and such that σ lifts to a curve $\tilde{\sigma}$ in $\tilde{M}$ which develops to an open arc with endpoints at $\mathrm{Fix}(A)$. (This latter condition will be valid under many interesting hypotheses, e.g., if φ_M is an isomorphism $\pi_1(M) \to \Gamma$, where Γ is a quasi-Fuchsian subgroup of G.) Let T_A denote the Hopf torus with holonomy A. Then we may find curves σ_M, σ_T on M and on T_A respectively with isomorphic neighborhoods. The sum $M + T_A$ along the σ-curves is a new $\mathbf{CP}^1$-manifold, the *graft* of M along A, topologically the union of M with an annulus $T_A|A$. Note that the graft of M along A is uniquely determined upon specifying $A \subset M$. We denote this graft by $M[A]$.

In the graft $M[A]$ there are curves corresponding to A. Grafting $M[A]$ along the new curve A is equivalent to summing $M|A$ with two copies of $T_A|A$, which is equivalent to summing $M|A$ with $T_A^{(2)}|A$. Thus we may inductively graft along the same curve: define $M[nA] = M[(n-1)A][A]$ so that n-fold iterated graft $M[nA] = M[A]\cdots[A] = M|A + T_A^{(n)}|A$.

More generally suppose that A and B are two disjoint simple closed curves satisfying our basic hypothesis that they have lifts which are arcs between the two hyperbolic fixed points of their holonomy. Then it is clear that in the graft $M[A]$ there is a curve corresponding to B and a curve corresponding to A in $M[B]$ such that the iterated grafts $M[A][B]$ and $M[B][A]$ are isomorphic $\mathbf{CP}^1$-manifolds. In this way we define $M[A, B]$ and $M[A_1, A_2, \cdots, A_n]$, where $\{A_1, A_2, \cdots, A_n\}$ is any set of disjoint simple closed curves satisfying the basic hypothesis.

1.4. Definition. Let $\mathbf{S}$ denote the set of all disjoint unions of homotopically nontrivial simple closed curves. A *grafting data* on M is defined to be a disjoint union of admissible simple closed curves on M. Define $\mathbf{S}_M \subset \mathbf{S}$ to be the set of all grafting data. If $\sigma \in \mathbf{S}_M$, then the *σ-graft* of M is defined to be $M[\sigma] = M[A_1, \cdots, A_n]$ where $A_1, \cdots, A_n$ represent the components of σ.

1.5. Definition. Suppose that S is a closed surface. We say that a homomorphism $\varphi : \pi_1(S) \to G$ is *Fuchsian* (respectively *quasi-Fuchsian*) if and only if φ is injective and its image is a Fuchsian (resp. quasi-Fuchsian) subgroup of G. If φ is Fuchsian (resp. quasi-Fuchsian) then a *Fuchsian* (resp. *quasi-Fuchsian*) *structure with holonomy* φ is one of the two $\mathbf{CP}^1$-manifolds of the form $\Omega/\varphi(\pi)$, where Ω is one of the two components of discontinuity of $\varphi(\pi)$.

1.6. Theorem C. *Let M be a $\mathbf{CP}^1$-manifold whose holonomy $\varphi : \pi_1(M) \to \Gamma \subset G$ is Fuchsian (respectively quasi-Fuchsian). Then there exists a disjoint union of nontrivial simple closed curves $\sigma \in \mathbf{S}$ and a Fuchsian (resp. quasi-Fuchsian) structure M_φ with holonomy φ such that $M = M_\varphi[\sigma]$.*

1.7. It is easy to reduce Theorem C to the Fuchsian case, because any quasi-Fuchsian $\varphi : \pi_1(M) \to \Gamma \subset G$ is quasiconformally conjugate to a Fuchsian representation $\varphi_0 : \pi_1(M) \to \Gamma \subset G$. That is, there exists a quasiconformal homeomorphism $h : \mathbf{CP}^1 \to \mathbf{CP}^1$ such that for all $\gamma \in \pi_1(M)$, $h\varphi h^{-1} = \varphi_0$. It is easy to see that Theorem C for φ is equivalent to Theorem C for φ_0; hence it suffices to consider the case that φ is Fuchsian.

Suppose that $\varphi : \pi \to \mathrm{PSL}(2, \mathbf{R}) \subset G$ is Fuchsian. We shall break the hypothesis that φ is Fuchsian into two parts: first, φ is a *real* representation since $\varphi(\pi_1(M)) \subset \mathrm{PSL}(2, \mathbf{R})$; second a real representation is Fuchsian if and only if the absolute value of its Euler class is maximized among all real representations (see [7] or [8]). We shall begin by exploring the consequences of the holonomy representation being real.

2. $\mathbf{CP}^1$-structures with real holonomy

2.1. We shall consider the decomposition $\mathbf{CP}^1 = H_+ \cup \mathbf{RP}^1 \cup H_-$, where the upper and lower hemispheres H_+ and H_- are defined by $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$, respectively. Since the sets H_+, $\mathbf{RP}^1$, H_- are each $\mathrm{PSL}(2, \mathbf{R})$-invariant, this decomposition passes to a π-invariant decomposition of the universal cover $\tilde{M}$. Namely, $\tilde{M} = \tilde{M}_+ \cup \tilde{M}_{\mathbf{R}} \cup \tilde{M}_-$, where $\tilde{M}_+ = \mathrm{dev}^{-1}(H_+)$, etc. Let $p : \tilde{M} \to M$ denote the projection, and write $M_+ = p(\tilde{M}_+)$, $M_{\mathbf{R}} = p(\tilde{M}_{\mathbf{R}})$, $M_- = p(\tilde{M}_-)$. Define the *$\mathbf{R}$-decomposition*: $M = M_+ \cup M_{\mathbf{R}} \cup M_-$. Let $\mathscr{G}_{\mathbf{P}}$ denote the symmetric 2-tensor on $H_\pm = \mathbf{CP}^1 \setminus \mathbf{RP}^1$ which restricts to the Poincaré metric on H_+ and to the pull-back of the Poincaré metric by complex conjugation on H_-. There is a unique symmetric 2-tensor $\mathscr{G}_M$ on the open subset $M_\pm = M_+ \cup M_- = M \setminus M_{\mathbf{R}} \subset M$ such that $p^*(\mathscr{G}_M) = \mathrm{dev}^*(\mathscr{G}_{\mathbf{P}})$.

2.2. Theorem. *Each component of M_+ or M_- is a complete hyperbolic surface, with ideal boundary a union of components of $M_{\mathbf{R}}$.*

Proof. On each component M_i of $M_\pm$, $\mathcal{G}_M$ restricts to a Riemannian metric g_i, for which the restrictions of p and dev are local isometries. Thus $(M_i, \mathcal{G}_i)$ is a hyperbolic 2-manifold. We claim that M_i is complete as a metric space. Suppose that $M_i \subset M_+$. To this end let $\{y_1, y_2, \cdots, y_k, \cdots\}$ be a Cauchy sequence in M_i; we will show it converges in M_i. Since M is compact the sequence $\{y_n\}$ has an accumulation point $z \in M$. We claim that $z = \lim y_n \in M_i$.

Lift the Cauchy sequence $\{y_n : n = 1, 2, 3, \cdots\}$ in M_i to a Cauchy sequence $\{\tilde{y}_n : n = 1, 2, 3, \cdots\}$ in $\tilde{M}_i$. Let $\tilde{z} \in \tilde{M}$ be the corresponding lift of z, which is a cluster point of the $\{\tilde{y}_n\}$. Since $\mathcal{G}_{\mathbf{P}}$ has a pole at $\mathbf{RP}^1$, and $\{\text{dev}(\tilde{y}_n)\}$ is Cauchy, it follows that $\text{dev}(\tilde{z}) \notin \mathbf{RP}^1$. Since $\text{dev}(\tilde{z})$ lies in the closure of H_+, it follows that $\text{dev}(\tilde{z}) \in H_+$. Hence $z \in M_i$.

There exists a neighborhood U of $\tilde{z} \in \tilde{M}_i$ such that $\text{dev}|_U$ is an isometry. For n sufficiently large, $\tilde{y}_n \in U$ so we may assume that $\tilde{y}_n \in U$ for all n. Since $\text{dev}|_U$ is an isometry, and $\{\text{dev}(\tilde{y}_n)\}$ is a Cauchy sequence in H_+, $\tilde{y}_n \to \tilde{z}$.

Since $\text{dev} : \tilde{M} \to \mathbf{CP}^1$ is a local diffeomorphism and each component of $H_\pm$ is an open 2-disk with boundary $\mathbf{RP}^1$, it follows that each M_i has boundary a union of components of $M_{\mathbf{R}}$. Since each component of $H_\pm$ has ideal boundary $\mathbf{RP}^1$, each M_i has ideal boundary a union of components of $M_{\mathbf{R}}$. This completes the proof of the lemma.

2.3. Remark. This decomposition is also discussed by Faltings [3], from a more conformal-geometric point of view. The approach taken here is more directly based on hyperbolic geometry.

It is worthwhile to point out the basic structure of a complete hyperbolic surface M_i with ideal boundary. Inside M_i there is a unique maximal convex subsurface such that the inclusion into M is a homotopy equivalence. Such a subsurface is called the *convex core* of M; we denote it by $\text{core}(M)$. Its boundary consists of closed geodesics. Furthermore we can see that the ends of M_{ii} are annuli of infinite area. The complement $M_i \setminus \text{core}(M_i)$ consists of annuli each bounded by a closed geodesic and an ideal boundary curve. The complement of the ideal set is a disjoint union of annuli we call *complete collars* about their bounding geodesics. These complete collars are natural representatives of the ends of M_i. There is a canonical deformation retraction $M \to \text{core}(M)$, along geodesics orthogonal to $\partial\text{core}(M)$.

If $\chi(M_i) = 0$, then M_i is topologically an annulus and the convex core of M_i is an essential simple closed geodesic γ_i. Then M_i is the union of two complete collars of the geodesic along their common geodesic boundary component. Such a hyperbolic 2-manifold we shall call a *complete tube* about γ_i.

2.4. First we observe some elementary topological properties of the **R**-decomposition of M:

Lemma. *Suppose M is closed and connected. Then no component of $M_{\pm}$ is simply connected.*

Proof. Suppose M_i is a simply connected component of $M_{\pm}$. Then the developing map determines a projective isomorphism $M_i \to H_{\pm}$. The ideal boundary of M_i is a component $M_{\mathbf{R}}^{(k)}$ of $M_{\mathbf{R}}$ whose developing map $\operatorname{dev}: \partial M_i \to \mathbf{RP}^1$ is a bijection. Let M_j be the other component of $M_{\pm}$ bounded by $M_{\mathbf{R}}^{(k)}$; then $M_i \cup M_{\mathbf{R}}^{(k)} \cup M_j$ is a union of two 2-disks along a common boundary and hence is an embedded open 2-sphere in M. This contradicts M being connected.

2.5. Corollary. *For each component M_i of $M_{\pm}$, the Euler characteristic $\chi(M_i) \leqslant 0$.*

2.6. The components satisfy $\chi(M_i) < 0$ except for certain annuli M_i. These annuli are quotients of $H \cup (\mathbf{RP}^1 \setminus \operatorname{Fix}(A))$ by the cyclic group of projective transformations generated by A. Note that if M is compact, then A will be strictly hyperbolic. There exists a closed geodesic c_i in M_i and a retraction $r: M_i \to c_i$ (given by orthogonal projection along geodesics) exhibiting M_i as a trivial **R**-bundle over c_i. These are the only components M_i with $\chi \geqslant 0$.

2.7. Let $\varphi: \pi \to \mathrm{PSL}(2, \mathbf{R})$ be a real representation. Let H_φ denote the oriented H_+-bundle over S with holonomy φ. (Recall that H_φ is defined as the quotient of the product H_+-bundle over the universal covering $\tilde{M}$ of M by the action of π which is by deck transformations on $\tilde{M}$ and by φ on the fiber.) Since the action by π on $\tilde{M}$ is proper with quotient M, the map $H_\varphi \to M$ induced by the projection $\tilde{M} \times H_+ \to \tilde{M}$ is a $\mathbf{CP}^1$-fibration. Let $e(\varphi) = e(H_\varphi) \in H_2(M; \mathbf{Z}) \cong \mathbf{Z}$ denote the Euler class of this oriented 2-disk bundle.

Lemma. *Let M be a connected closed $\mathbf{CP}^1$-manifold with holonomy $\varphi: \pi \to \mathrm{PSL}(2, \mathbf{R})$. Let $M = M_+ \cup M_{\mathbf{R}} \cup M_-$ be the **R**-decomposition of M. Then $e(\varphi) = \chi(M_+) - \chi(M_-)$.*

Proof. Let $\mathbf{P}_\varphi$ denote the flat $\mathbf{CP}^1$-bundle with holonomy φ. Then the bundle $\mathbf{P}_\varphi$ decomposes as $H_{+\varphi} \cup \mathbf{RP}_\varphi \cup H_{-\varphi}$, where $H_{+\varphi}$ and $H_{-\varphi}$ are oriented 2-disk bundles and $\mathbf{RP}_\varphi$ is an oriented $\mathbf{RP}^1$-bundle. Define a bundle map $F: \mathbf{P}_\varphi \to H_\varphi$ by requiring that it be on each fiber the identity map $H_+ \subset H_+$ or reflection $H_- \to H_+$ in $\mathbf{RP}^1$ (i.e., complex conjugation). Let $D: M \to \mathbf{P}_\varphi$ be the developing section of M. The composite $F \circ D$ is a section of H_φ.

In order to compute the Euler class of H_φ, we consider the self-intersection number of the section $f = F \circ D$. We shall compute the self-intersection number of f from the self-intersection number of the developing section D in

$\mathbf{P}_\varphi$. By general position, we may assume the self-intersections of D are disjoint from the subbundle $\mathbf{RP}_\varphi$. Furthermore a tubular neighborhood of D is equivalent to the tangent disk-bundle (see Goldman [7]). Thus the Euler class $e(\varphi)$ has contributions from M_+, which sum to $\chi(M_+)$ and contributions from M_-, which sum to $\chi(M_-)$.

The Euler class $e(\varphi)$ can then be computed as the self-intersection number of $F \circ D$. The computations are exactly the same as for the self-intersection number of D, except that the contributions from M_- are counted negatively: the total self-intersection number of $F \circ D$ equals (contributions from M_+) $-$ (contributions to M_-) which equals $\chi(M_+) - \chi(M_-)$. This proves Lemma 2.7.

2.8. To prove the Fuchsian surgery Theorem C we shall need the following, which is discussed in Goldman [7]:

Corollary. *Let* $\varphi \in \operatorname{Hom}(\pi, \mathrm{PSL}(2,\mathbf{R}))$ *be Fuchsian. Then* $e(\varphi) = \pm\chi(M)$.

Proof. Apply §2.7 to the developing section of the Fuchsian structure with holonomy φ. Either M_+ or M_- is empty, so $e(\varphi) = \pm\chi(M)$.

Remark. Although §2.8 is obtained here as a corollary of §2.7, its proof is really implicit in the proof of §2.7, as a key special case.

2.9. *Conclusion of the proof of Theorem* C. Suppose M is a closed $\mathbf{CP}^1$-manifold with holonomy homomorphism $\varphi: \pi \to \mathrm{PSL}(2,\mathbf{R})$ which is Fuchsian. By possibly changing the orientation on M, we shall assume that $e(\varphi) = \chi(M)$. Let $M = M_+ \cup M_{\mathbf{R}} \cup M_-$ be the $\mathbf{R}$-decomposition of M. Now

$$\chi(M_+) + \chi(M_-) = \chi(M) = e(\varphi) = \chi(M_+) - \chi(M_-),$$

whence $\chi(M_-) = 0$.

Write M_- as a union of components $M_1 \cup \cdots \cup M_k$. By 2.5 each M_i has nonpositive Euler characteristic. It follows that each $\chi(M_i) = 0$.

Thus each component of M_- is a complete tube (i.e., an annulus $H_-/\langle A\rangle$ as described above in §2.3) with ideal boundary $(\mathbf{RP}^1 \setminus \mathrm{Fix}(A))/\langle A\rangle$. For a given boundary component $M_{\mathbf{R}(k)}$ of M_i, let M_j denote the component of M_+ whose boundary contains $M_{\mathbf{R}(k)}$. Then there is a complete collar in M_j which joins M_i at $M_{\mathbf{R}(k)}$. Similarly in the other component of M_+ which is adjacent to M_i there is a complete collar which meets one on M_i. The union of M_i with these two complete collars is a split Hopf torus $T_A|A$, which is easily seen to be a summand of the $\mathbf{CP}^1$-manifold M.

This decomposes M into a union of disjoint compact surfaces with boundary satisfying $\chi(M_i) \leqslant 0$. The complement of the union of the complete tubes and the complete collars is a convex hyperbolic surface with a set of boundary identifications which yield the Fuchsian $\mathbf{CP}^1$-manifold M_0. The components which are annuli determine grafting data σ for which $M = M_0[\sigma]$, where M_0 is the Fuchsian structure with holonomy φ.

2.10. Construction of $\mathbf{CP}^1$-structures with real holonomy. We have seen how a $\mathbf{CP}^1$-structure with real holonomy may be considered a union of two (not necessarily connected) hyperbolic surfaces along geodesic boundary. We will see how to use this theorem to explicitly construct projective structures with various kinds of holonomy homomorphisms. In particular we shall express the inverse images under

$$\mathrm{hol} : \mathbf{CP}^1(S) \to \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{C}))/\mathrm{PSL}(2, \mathbf{C})$$

of the subsets

$$\mathscr{T}_S \subset \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R}) \subset \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{C}))/\mathrm{PSL}(2, \mathbf{C}).$$

Here $\mathscr{T}_S$ denotes the subset of $\mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R})$ consisting of equivalence classes of orientation-preserving Fuchsian representations. We will see that $\mathrm{hol}^{-1}(\mathscr{T}_S)$ consists of all the integral points in $\mathscr{T}_S \times \mathscr{ML}_S$ and $\mathrm{hol}^{-1}(\mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R}))$ consists of the half-integral points in $\mathscr{T}_S \times \mathscr{ML}_S$.

Let $M = M_+ \cup M_-$ be the $\mathbf{R}$-decomposition of M. Let $\{M_i : i \in I\}$ denote the set of components of $M_\pm$. Consider the union $M_0 = \bigcup\{\mathrm{core}(M_i) : i \in I\}$. Its complement in M consists of annuli A_γ bounded by geodesic arcs γ all of which have the property that their two bounding circles develop to the same semicircular arc $\tilde{\gamma}$.

The annuli A_γ in the complement of the convex cores all possess a retraction onto the closed geodesics γ which they determine. Thus there is a canonical identification map of M_0 onto a connected hyperbolic surface M_0'. This map $M \to M_0'$ is an isometry with folds along closed geodesics corresponding to various γ.

For example, in the Schottky uniformization of genus 2, the hyperbolic surface M_0' is a surface of genus 2, realized as the double of either (a) a pair of pants with geodesic boundary or (b) a torus with one geodesic boundary component. In any structure with Fuchsian holonomy, M_0' is just the Fuchsian structure.

2.11. The decomposition into annuli and convex hyperbolic surfaces is a special case of a remarkable parametrization of $\mathbf{CP}^1$-structures due to Thurston (unpublished) (see also Epstein-Marden [2]). Thurston shows that a $\mathbf{CP}^1$-manifold may be viewed as a hyperbolic 2-manifold "bent" in a locally convex way inside hyperbolic 3-space. Thurston's identification can be succinctly stated in terms of deformation spaces as follows.

Theorem (*Thurston*). *There is a canonical bijection* $\Theta : \mathbf{CP}^1(S) \to \mathscr{T}_S \times \mathscr{M}\mathscr{L}_S$, *where* $\mathbf{CP}^1(S)$ *is the deformation space of* (*homotopy*) $\mathbf{CP}^1$*-structures on* S, $\mathscr{T}_S$ *is the Teichmuller space of* S, *and* $\mathscr{M}\mathscr{L}_S$ *is the Thurston space of measured geodesic laminations of* S.

Under various assumptions on the holonomy representation, the Thurston parameters will be extremely easy to calculate (unlike the classical conformal parameters). However, it is important to point out that the hyperbolic structure on M given by Thurston's correspondence is generally not the one produced by the uniformization theorem: Thurston's metric is generally conformally inequivalent to the Poincaré metric on M.

We shall only describe the part of Thurston's theorem which deals with laminations of *rational slope*, i.e., laminations supported on a closed one-dimensional submanifold. In that case Thurston's bending construction can be neatly described by the insertion of θ-annuli into Fuchsian structures. We shall henceforth assume M is a closed hyperbolic surface. We presently define two kinds of $\mathbf{CP}^1$-manifolds, θ-crescents and θ-annuli.

2.12. Definition. Let $\theta > 0$ be any positive real number, and let W_θ be the closed region in $\mathbf{C}$ consisting of all complex numbers z such that $0 \leqslant \operatorname{Im} z \leqslant \theta$. Consider the $\mathbf{CP}^1$-structure induced on W_θ by the exponential map $\exp : \mathbf{C} \to \mathbf{C}^*$. We shall denote the resulting $\mathbf{CP}^1$-manifold by C_θ. A *θ-crescent* is any $\mathbf{CP}^1$-manifold C with boundary projectively equivalent to C_θ. If $f : C_0 \to C$ is such a projective map between θ-crescents, we say that the *vertices* of C are the images $f(0)$, $f(\infty)$. Suppose that $A \in \mathrm{PSL}(2, \mathbf{C})$ leaves invariant a θ-crescent C. Then there is an induced projective automorphism $\hat{A} : C \to C$. The resulting quotient $\mathbf{CP}^1$-manifold is a *θ-annulus*.

If $0 < \theta < 2\pi$, then a θ-crescent is nothing more than a region bounded by circular arcs which intersect at angle θ. Two such θ-crescents are adjacent if they share a common edge. Clearly if C_1 and C_2 are θ_1- and θ_2-crescents, respectively, and are adjacent, then the union $C_1 \cup C_2$ is $\theta_1 + \theta_2$-crescent.

Note that if $\tau_i / \langle A \rangle$, $i = 1, 2$, are θ_i-annuli covered by adjacent crescents, then their union is canonically a θ-annulus.

If τ is a θ-crescent bounded by circular arcs $\partial_+(\tau)$, $\partial_-(\tau)$, let $e(\tau)$ denote the unique elliptic transformation fixing the vertices of τ and taking $\partial_+(\tau)$ to $\partial_-(\tau)$.

2.13. Let M_0 be a closed hyperbolic surface. We shall describe how to "bend" M_0 along a measured geodesic lamination μ supported on a disjoint union of closed geodesics. Then μ consists of disjoint simple closed curves γ_i in M together with positive weights m_i attached to them. We may write

$$\mu = \sum m_i \gamma_i .$$

To "bend" the structure along μ, we shall enlarge M_0 by first splitting M_0 along the curves γ_i, then inserting θ_i-annuli along the boundary curves γ_i^+, γ_i^-, where $\theta_i = 2\pi m_i$.

The developing map (and hence the structure) can be constructed as follows. Let $\{M_i : i \in I\}$ denote the components of $M | \bigcup\{\gamma_i : i \in I\}$. For each M_i we can find a fundamental domain $F_i \subset \tilde{M}_i \subset \tilde{M}$. Furthermore these fundamental domains can be so chosen that the union $F = \bigcup\{F_i : i \in I\}$ is a fundamental domain for π acting on $\tilde{M}$.

Choose a germ of a projective immersion $S \to \mathbf{CP}^1$ near $x \in S$. We may assume that $x \notin \mathrm{int}(M_i)$ for some $i \in I$. Then one can find a uniquely determined developing map for the fundamental domain $F_{(0)}$ for this M_i. On the sides of $F_{(0)}$ which correspond to curves γ_i in the lamination, we attach θ_i-crescents. The fundamental domain for the region M_j adjacent to M_i along a γ is then immersed by $e(\tau_\gamma) \circ \mathrm{dev}_0$, where dev_0 is the original developing map restricted to F_j. Continuing in this manner one eventually defines the entire developing map by requiring dev to be π-equivariant.

2.14. Our primary use of Thurston's correspondence is to neatly characterize the projective structures with respectively Fuchsian and real holonomy.

Theorem. *Let* $\mathrm{hol}: \mathbf{CP}^1(S) \to \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{C}))/\mathrm{PSL}(2, \mathbf{C})$ *be the map which assigns to a homotopy* $\mathbf{CP}^1$*-manifold the equivalence class of its holonomy homomorphism. Let*

$$\mathscr{T}_S \subset \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R}) \subset \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{C}))/\mathrm{PSL}(2, \mathbf{C})$$

be the natural map (*where* $\mathscr{T}_S$ *is embedded in* $\mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R})$ *as a connected component.*

(i) *Let* $\mathrm{hol}^{-1}(\mathscr{T}_S)$ *denote the subset of* $\mathbf{CP}^1(S)$ *consisting of* $\mathbf{CP}^1$*-manifolds with Fuchsian holonomy. Then* θ *defines a bijection between* $\mathrm{hol}^{-1}(\mathscr{T}_S) \to$ *and* $\mathscr{T}_S \times \mathscr{ML}_S(\mathbf{Z})$.

(ii) *Let* $\mathrm{hol}^{-1}(\mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R})))/\mathrm{PSL}(2, \mathbf{R})$ *denote the subset of* $\mathbf{CP}^1(S)$ *comprising* $\mathbf{CP}^1$*-manifolds with real holonomy. Then* Θ *defines a bijection between* $\mathrm{hol}^{-1}(\mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R}))$ *and* $\mathscr{T}_S \times \mathscr{ML}_S(\frac{1}{2}\mathbf{Z})$.

In terms of the above terminology, Theorem C has the following interpretation: the general $\mathbf{CP}^1$-manifold with Fuchsian holonomy is obtained by inserting 2π-annuli into the Fuchsian structure M_0. Thus the subset of $\mathbf{CP}^1(S)$ comprising $\mathbf{CP}^1$-manifolds with Fuchsian holonomy is precisely $\Theta^{-1}(\mathscr{T}_S \times \mathscr{ML}_S(\mathbf{Z}))$, where $\mathscr{ML}_S(\mathbf{Z})$ denotes the set of integral points in $\mathscr{ML}_S$, i.e. isotopy classes of disjoint unions of homotopically nontrivial simple closed curves.

Theorem 2.2 gives an interpretation of $\mathbf{CP}^1$-structures with real holonomy as a kind of union of complete hyperbolic structures with ideal boundary. By retracting these hyperbolic manifolds to their convex cores we obtain a hyperbolic surface M_0' which is the one specified by the Thurston isomorphism

$$\Theta : \mathbf{CP}^1(S) \to \mathscr{T}_S \times \mathscr{M}\mathscr{L}_S.$$

Moreover the bending lamination is easily seen to be the curves γ counted with multiplicity $m(\gamma)$ corresponding to the number of components of $M_\pm$ collapsed to γ. The bending parameters are easily seen to be equal to $\pi m(\gamma)$ around each γ. Identifying a measure 2π of bending to 1, we see that the subset of the measured lamination space $\mathscr{M}\mathscr{L}_S$ corresponding to $\mathbf{CP}^1$-structures with real holonomy is precisely the subset of half-integral points $\mathscr{M}\mathscr{L}_S(\frac{1}{2}\mathbf{Z})$.

When the measure around a curve γ is even, say $m(\gamma) = 2n$, then the bending parameter around γ is a multiple $2\pi n$ of 2π. In this case there are exactly n Hopf manifolds split along γ which become identified to γ. This corresponds to an n-fold graft along γ. Thus the integral points of $\mathscr{M}\mathscr{L}_S$ correspond to $\mathbf{CP}^1$-manifolds with Fuchsian holonomy.

When the measure around γ is odd, say $m(\gamma) = 2n + 1$, then we find exactly n split Hopf manifolds which map to γ. After removing them we find two components $M_+(\gamma) \subset M_+$, $M_-(\gamma) \subset M_-$ which are adjacent along γ. These two components have opposite orientation and we only obtain a hyperbolic structure on M when the developing map for $M_-(\gamma)$ is obtained from the developing map of $M_+(\gamma)$ by composition with a reflection in the geodesic γ. This is an example of a "hyperbolic structure with fold singularities" along γ, and one obtains a fold singularity along each curve γ for which there are an odd number of π-annuli collapsed to γ.

3. Some real projective structures on annuli and tori

3.1. The grafting process for $\mathbf{RP}^2$-structures is more complicated than for $\mathbf{CP}^1$-structures, even for surfaces of $\chi(S) = 0$. For the examples we consider the developing maps are classified by disjoint unions of families of simple closed curves weighted by elements of a certain semigroup. This semigroup is a certain class of $\mathbf{RP}^2$-structures on annuli with geodesic boundary. Once these $\mathbf{RP}^2$-structures are classified, it will be easy to adapt the proof of the Fuchsian surgery theorem for $\mathbf{CP}^1$-structures to $\mathbf{RP}^2$-structures. In this section we develop the notion of grafting $\mathbf{RP}^2$-manifolds, applying it to the classification of $\mathbf{RP}^2$-manifolds with holonomy a cyclic group of $\mathbf{R}$-hyperbolic projective maps.

Let M be an $\mathbf{RP}^2$-surface (i.e., a surface with $\mathbf{RP}^2$-structure). An arc in M is *geodesic* if it has a lift which is developed bijectively onto a line segment in $\mathbf{RP}^2$. M is a compact surface *with geodesic boundary* if and only if ∂M is a (disjoint) union of closed geodesics. Suppose that M_1 and M_2 are $\mathbf{RP}^2$-manifolds with geodesic boundary and $B_1 \subset \partial M_1$ and $B_2 \subset \partial M_2$ are unions of boundary components for which there exist neighborhoods N_1 and N_2 of B_1 and B_2 respectively and a projective isomorphism $j: N_1 \to N_2$. Then the *sum* of M_1 and M_2 (along $j: N_1 \to N_2$) is the $\mathbf{RP}^2$-manifold with geodesic boundary $M = M_1 + M_2$ given by the disjoint union $M_1 \cup M_2$ modulo the identifications given by j.

Definition. A *special* $\mathbf{RP}^2$*-annulus* is a compact $\mathbf{RP}^2$-manifold with geodesic boundary homotopy-equivalent to a circle.

This section will deal with the classification of special $\mathbf{RP}^2$-annuli and related structures on tori and Klein bottles whose holonomy group is a cyclic group generated by a projective transformation of $\mathbf{RP}^2$ represented by a diagonal matrix in $\mathrm{GL}(3, \mathbf{R})$ having distinct positive eigenvalues. This will be the only case needed for describing the Fuchsian surgery theorem for oriented $\mathbf{RP}^2$-structures. The general classification of $\mathbf{RP}^2$-structures on surfaces of zero Euler characteristic is given in [6], upon which this treatment is based.

3.2. We shall find it useful to fix the following notation concerning the group of projective transformations of $\mathbf{RP}^2$. The projective group consisting of all collineations of $\mathbf{RP}^n$ is the group $\mathrm{PGL}(n+1, \mathbf{R})$. If n is even, this group is isomorphic to $\mathrm{SL}(n+1, \mathbf{R})$. The composition of the inclusion $\mathrm{SL}(n+1, \mathbf{R}) \hookrightarrow \mathrm{GL}(n+1, \mathbf{R})$ with the projection $\mathrm{GL}(n+1, \mathbf{R}) \to \mathrm{PGL}(n+1, \mathbf{R})$ is an isomorphism $\mathrm{SL}(n+1, \mathbf{R}) \to \mathrm{PGL}(n+1, \mathbf{R})$ of real analytic groups; the inverse homomorphism $\mathrm{GL}(n+1, \mathbf{R}) \to \mathrm{SL}(n+1, \mathbf{R})$ is given by

$$A \mapsto \det(A)^{-1/(n+1)} A.$$

3.3. Classification of special $\mathbf{RP}^2$-annuli. Consider a diagonal matrix $A \in \mathrm{SL}(3, \mathbf{R})$

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

where $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Then $\lambda_1 \lambda_2 \lambda_3 = 1$. We shall identify A with the collineation of $\mathbf{RP}^2$ which it induces.

Corresponding to the three coordinate axes in $\mathbf{R}^3$ are the three stationary points p_1, p_2, p_3 for A acting on $\mathbf{RP}^2$. The stationary point p_1 corresponding to the largest eigenvalue is a repelling fixed point for A; similarly p_2 and p_3

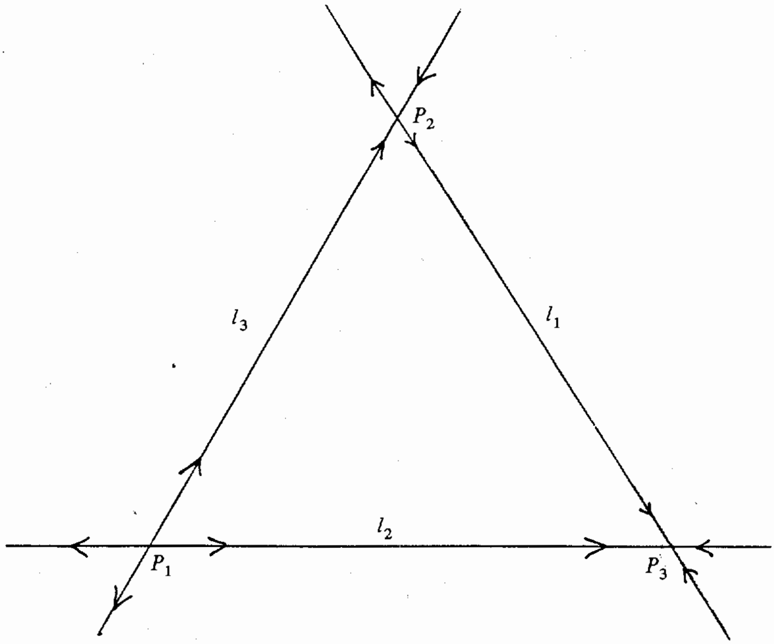

FIGURE 1

are respectively a saddle point and an attracting fixed point (compare Figure 1). The coordinate planes in $\mathbf{R}^3$ define invariant lines in $\mathbf{RP}^2$ joining the fixed points in pairs: we denote by l_1 the line containing p_2 and p_3, by l_2 the line containing p_1 and p_3, etc.

There are many special $\mathbf{RP}^2$-annuli with holonomy generated by A. We shall see their structure by decomposing them into eight types of building blocks. Let $l = \bigcup l_i$ be the union of invariant lines and let Δ_i, $i = 1, 2, 3, 4$, be the four components of the complement $\mathbf{RP}^2 \setminus l$. The interiors of these eight special $\mathbf{RP}^2$-annuli represent four equivalence classes, i.e. the quotients $\Delta_i/\langle A\rangle$, $i = 1, 2, 3, 4$.

3.4. Definitions. An *elementary annulus* is a special $\mathbf{RP}^2$-annulus whose interior is a quotient of a triangular domain Δ_i. Define a *boundary label* to be a pair of distinct lines in l, and a *development label* to be one of the triangular regions Δ_i, $i = 1, 2, 3, 4$. *The elementary annulus with holonomy A with labels* $(\Delta_i, l_{j,k})$ is the quotient of $\Delta_i \cup (u(l_j) \cup u(l_k))$ by A where $u(l_n)$ denotes the component of $l \setminus \mathrm{Fix}(A)$ on l_n bounding Δ_i. We shall denote this $\mathbf{RP}^2$-annulus by $E^i_{j,k}(A)$ (see Figure 2).

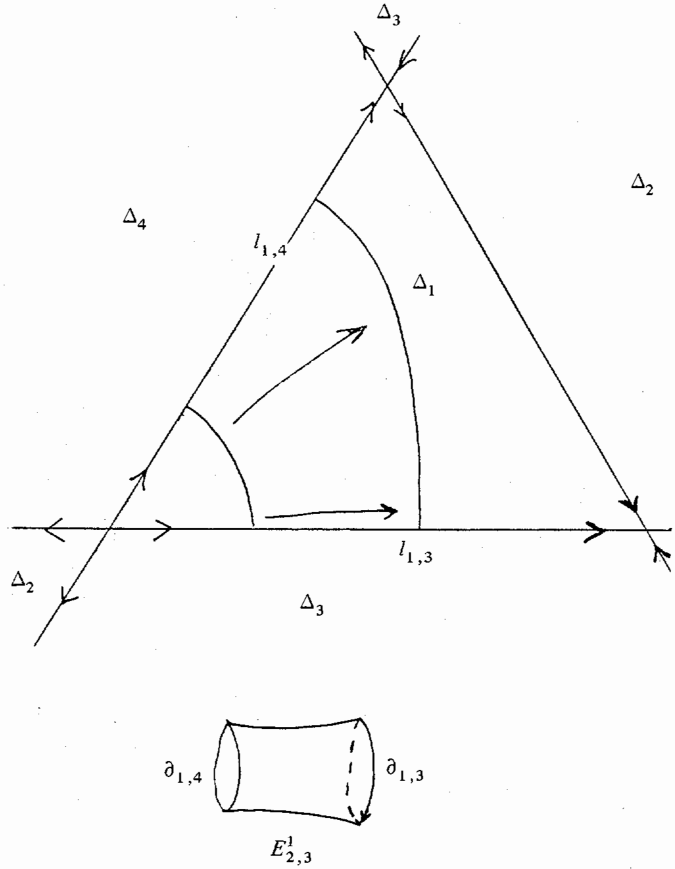

FIGURE 2. An elementary $\mathbf{RP}^2$-annulus

Note that $\langle A \rangle$ does not act properly on $\mathbf{RP}^2 \setminus (p_2 \cup l_2)$ so that $l_{1,3}$ and $l_{3,1}$ are not allowed as boundary labels. Thus we may construct many new special $\mathbf{RP}^2$-annuli with holonomy A by gluing together various elementary annuli along common boundary.

3.5. Lemma. *Let $\Delta_M = p(\mathrm{dev}^{-1}(\mathbf{RP}^2 \setminus l))$ and let $V = \bigcup V_j$, where V_j denotes the subset $p(\mathrm{dev}^{-1}(l_j))$ of M. Let B be a component of Δ_M. Then B is an elementary annulus with boundary a pair of components of V.*

Proof. There is an A-invariant Riemannian metric $\mathcal{G}_\Delta$ on $\mathbf{RP}^2 \setminus l$. In affine coordinates on $\mathbf{RP}^2 \setminus l_1$, for example, we may take $\mathcal{G}_\Delta = x_2^{-2}(dx_2)^2 + x_3^{-2}(dx_3)^2$. There is a unique metric $\mathcal{G}_M$ on Δ_M such that $p^*\mathcal{G}_M = \mathrm{dev}^*\mathcal{G}_\Delta$. By the same argument as 2.2, $\mathcal{G}_M$ is a complete metric on Δ_M. It follows that dev restricted to $\tilde{B}$ is a local isometry onto one of the components Δ of $\mathbf{RP}^2 \setminus l$. It follows that $\mathrm{dev}: \tilde{B} \to \Delta$ is a covering map and hence a homeomorphism.

3.6. Corollary. *Let M be a $\mathbf{RP}^2$-annulus. Then there exist elementary annuli M_i ($i = 1, \cdots, n$) such that $M = M_1 + \cdots + M_n$.*

Using this decomposition into elementary pieces, it is easy to classify $\mathbf{RP}^2$-annuli. Each M_i is bounded by two geodesics, one from V_2 and one from $V_1 \cup V_3$. Let γ be a simple arc on M with endpoints in each of the two components of M. By general position we assume that γ is transverse to the decomposition curves V. Furthermore we shall take γ so as to have a minimal number of intersections with V. Record the sequence of V_i which γ intersects. This sequence is subject to two constraints: (i) no V_i may follow itself; (ii) V_1 and V_3 may never follow each other. Conversely given a sequence of elements of $\{V_1, V_2, V_3\}$ satisfying (i) and (ii), there corresponds a special $\mathbf{RP}^2$-annulus with holonomy A.

3.7. The basic surgery process is, as with $\mathbf{CP}^1$-structures, to insert annuli into projective structures split along nontrivial curves. In order that the holonomy be unchanged during this modification we shall want to insert special $\mathbf{RP}^2$-annuli whose boundary components lie in the same V_i, and moreover develop to the same component of $l_i \setminus \mathrm{Fix}(A)$. (Indeed we will require that the boundary components lie in V_2.) Thus we shall need to consider those special $\mathbf{RP}^2$-annuli M such that there is a projective quotient map $M \to M'$ (where M' is a closed $\mathbf{RP}^2$-manifold) which identifies boundary components.

Definition. A special $\mathbf{RP}^2$-*torus* (resp. an $\mathbf{RP}^2$-*Klein bottle*) is a closed orientable $\mathbf{RP}^2$-manifold M homeomorphic to a torus (resp. a Klein bottle) which possesses a simple closed geodesic C such that $M|C$ is a special $\mathbf{RP}^2$-annulus.

It is an elementary matter now to classify the various developing maps of special $\mathbf{RP}^2$-tori and special $\mathbf{RP}^2$-Klein bottles. Split M along a component C of V_2 to obtain a special $\mathbf{RP}^2$-annulus with holonomy $\langle A \rangle$. The projective equivalence class of $M|C$ is described by a sequence of symbols $\{V_1, V_2, V_3\}$ satisfying the two conditions (i), (ii) above. Furthermore this sequence both begins and ends with V_2.

Conditions (i) and (ii) imply that this sequence takes the form $V_2 V_{i_1} \cdot V_2 \cdots V_2 V_{i_n} \cdot V_2$ where the $i_1, \cdots, i_n$ are either 1 or 3. Thus we may rewrite this sequence (after dropping the terminal V_2) as a *positive* word $w(x, y)$ (i.e. all exponents positive) in symbols x, y where x represents $V_2 V_3$ and y represents $V_2 V_1$.

In order that the special $\mathbf{RP}^2$-annulus can be identified to become a torus or Klein bottle, it is necessary that the two boundary components develop to the

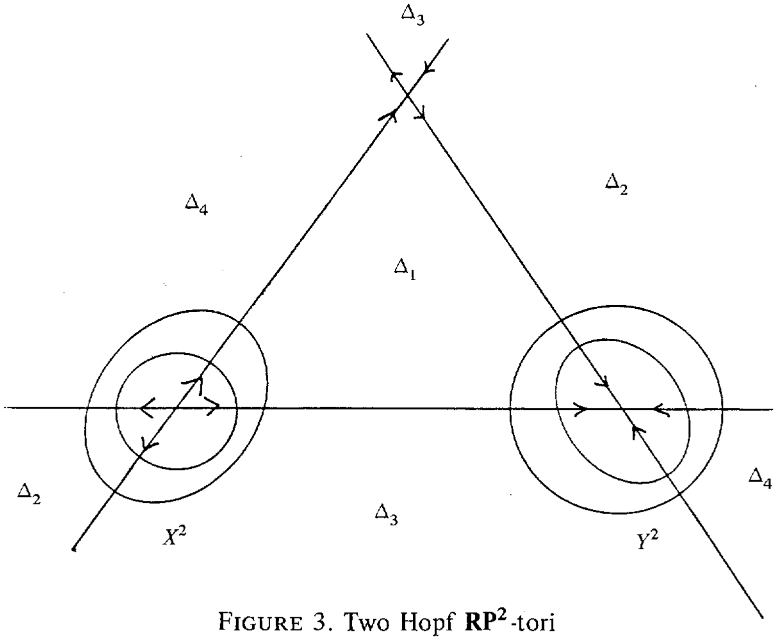

FIGURE 3. Two Hopf $\mathbf{RP}^2$-tori

same component of $l_2 \setminus \{p_1, p_3\}$. It is easy to see that this is equivalent to the condition that the total exponent sum of w in x and y be even. In that case we shall say $w(x, y)$ is an *even word*: otherwise $w(x, y)$ is an *odd word*.

Suppose M is a special $\mathbf{RP}^2$-annulus whose boundary components develop to the same component of $l_2 \setminus \{p_1, p_3\}$. As above M determines a positive word $w(x, y)$. The condition that the special $\mathbf{RP}^2$-annulus identifies to a torus and not a Klein bottle is a stronger "evenness" condition on $w(x, y)$ obtained as follows. Let $\gamma \subset M$ be a curve joining the two boundary components, which intersects each component of each V_i exactly once when the special $\mathbf{RP}^2$-annulus $A(w)$ determined by $w(x, y)$ identifies to a torus if and only if γ is orientation-preserving. (Compare Figure 3.)

3.8. Lemma. *γ is orientation-preserving if and only if the x-exponent sum (and hence the y-exponent sum) of w is even.*

Proof. Let γ' denote the closed loop in $\mathbf{RP}^2 \setminus \{p_1, p_2, p_3\}$ to which γ identifies. Since $w(x, y)$ is even, we may write $w(x, y) = w'(x^2, xy, yx, y^2)$. Let $\beta_1, \beta_2, \beta_3, \beta_4$ denote the elements of $\pi_1(\mathbf{RP}^2 \setminus \{p_1, p_2, p_3\})$ corresponding to x^2, xy, yx, y^2 respectively (compare Figure 4). The homotopy class $[\gamma'] \in \pi_1(\mathbf{RP}^2 \setminus \{p_1, p_2, p_3\})$ then admits the expression $[\gamma'] = w'(\beta_1, \beta_2, \beta_3, \beta_4)$. Of these four elements, β_1 and β_4 (corresponding to x^2 and y^2) preserve orientation and β_2 and β_3 (corresponding to xy and yx) reverse orientation. q.e.d.

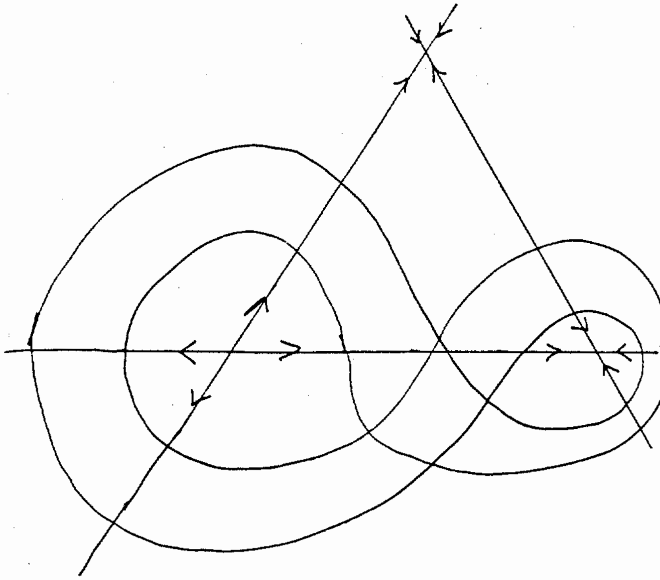

FIGURE 4. The $\mathbf{RP}^2$-torus X^2Y^2

We shall say that a positive word $w(x, y)$ is *completely even* if the exponent sums of $w(x, y)$ in both x and y are even. Otherwise we shall say that $w(x, y)$ is *partially odd*. Two words $w_1(x, y)$ and $w_2(x, y)$ are *cyclically equivalent* if there exists positive words $u(x, y)$, $v(x, y)$ such that $w_1(x, y) = u(x, y)v(x, y)$ and $w_2(x, y) = v(x, y)u(x, y)$. Together Corollary 3.6 and Lemma 3.8 imply:

3.9. Proposition. *Let A be a collineation of* $\mathbf{RP}^2$ *represented by a diagonalizable element of* SL(3; **R**) *having distinct positive eigenvalues. Let $\langle A\rangle$ denote the cyclic group generated by A.*

(i) *Let M be homeomorphic to a 2-torus and fix a surjective homomorphism $h:\pi_1(M) \to \langle A\rangle$. Then the set of projective equivalence classes of* $\mathbf{RP}^2$*-structures on M with holonomy h corresponds bijectively to the set of all cyclic equivalence classes of completely even positive words.*

(ii) *Let M be homeomorphic to a Klein bottle and fix a surjective homomorphism $h:\pi_1(T^2) \to \langle A\rangle$. Then the set of projective equivalence classes of* $\mathbf{RP}^2$*-structures on M with holonomy h corresponds bijectively to the set of all cyclic equivalence classes of partially odd even positive words.* (*If A is diagonalizable over* **R** *and has an odd number of negative eigenvalues, then the set of projective equivalence classes of* $\mathbf{RP}^2$*-structures on M with holonomy h corresponds bijectively to the set of cyclic equivalence classes of completely even positive words.*)

If M is an oriented special $\mathbf{RP}^2$-annulus for which there exists an identification map $M \to M'$, where M' is a special $\mathbf{RP}^2$-torus (i.e., the identification

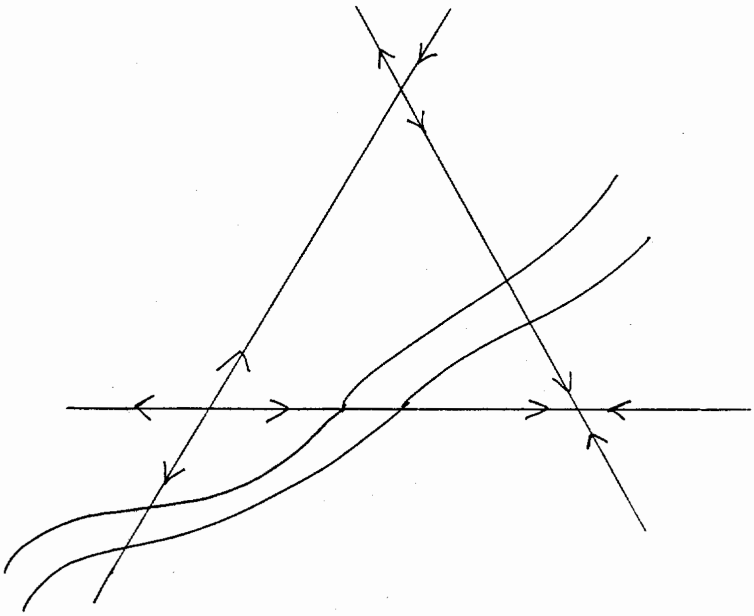

FIGURE 5. An $\mathbf{RP}^2$-Klein bottle XY

map is given by the identity map in local projective coordinates), then there is a well-defined completely even positive word $w(x, y)$ which describes its developing map. The various choices of geodesics on M' along which to split M' into a special $\mathbf{RP}^2$-annulus correspond to the various boundary components which develop to the same component of $l_2 \setminus \{p_1, p_3\}$. As above M determines an even word $w(x, y)$. But given a special $\mathbf{RP}^2$-torus M', there may be several completely even positive words corresponding to various ways to split M' into a special $\mathbf{RP}^2$-annulus; however all such words are cyclically equivalent. An example of such a Klein bottle is given in Figure 5.

4. $\mathbf{RP}^2$-manifolds with Fuchsian holonomy

In this section we shall prove the Fuchsian surgery theorem for $\mathbf{RP}^2$-manifolds. In this section M will be an $\mathbf{RP}^2$-manifold of negative Euler characteristic. For the sake of clarity we shall assume M is orientable (the results for nonorientable M can be easily deduced from the orientable case).

Let $C \supset \mathbf{RP}^2$ be a conic. Its stabilizer in $\mathrm{PGL}(3; \mathbf{R})$ is conjugate to $\mathrm{PO}(2,1) \subset \mathrm{PGL}(3; \mathbf{R})$. We shall say that a subgroup is *quadric* if it preserves a conic C. (This is the analogue of real holonomy for $\mathbf{CP}^1$-structures.) A representation

$\varphi:\pi \to \mathrm{PGL}(3;\mathbf{R})$ is *Fuchsian* if it is a faithful homomorphism onto a discrete quadric subgroup of $\mathrm{PGL}(3;\mathbf{R})$.

Suppose that $\varphi:\pi \to \mathrm{PSL}(2,\mathbf{R})$ is Fuchsian. Then there is a preferred $\mathbf{RP}^2$-manifold $M_0(\varphi)$ with holonomy φ. Namely, let W denote the convex region bounded by the conic C. Then $\Gamma = \varphi(\pi)$ is a discrete group which acts properly and freely on W. We refer to this structure as the Fuchsian structure (or the *convex structure*) with holonomy representation φ.

A Fuchsian $\mathbf{RP}^2$-manifold M is *convex*: every arc in M is homotopic to a geodesic arc keeping endpoints fixed. Equivalently, M is the quotient of a convex domain Ω in $\mathbf{RP}^2$ by a discrete group of projective transformations acting properly and freely. By analogy with the complex projective case, we say a representation $\varphi:\pi \to \mathrm{PGL}(3;\mathbf{R})$ is $\mathbf{RP}^2$*-quasi-Fuchsian* if it arises as the holonomy representation of a convex $\mathbf{RP}^2$-structure.

We shall describe a procedure for constructing all possible $\mathbf{RP}^2$-manifolds with $\mathbf{RP}^2$-quasi-Fuchsian holonomy. We shall express the general one as a graft of the convex structure.

4.1. For some basic properties on convex $\mathbf{RP}^2$-structures we refer to Kuiper [16] and Benzecri [1]. In particular Kuiper proved that if a domain Ω covers a compact convex $\mathbf{RP}^2$-manifold then either $\partial\Omega$ is a conic or $\partial\Omega$ fails to be C^2 on a dense subset. Moreover Benzecri [1] prove that, under the same hypotheses on Ω, either Ω is a triangle or $\partial\Omega$ is always C^1. When $\chi(M) < 0$, then Ω is actually strictly convex (Kuiper). For more examples of convex $\mathbf{RP}^2$-structures see Vinberg [25], Vinberg-Kac [13], Goldman [6]. Figure 6 illustrates one of the convex domains which may arise.

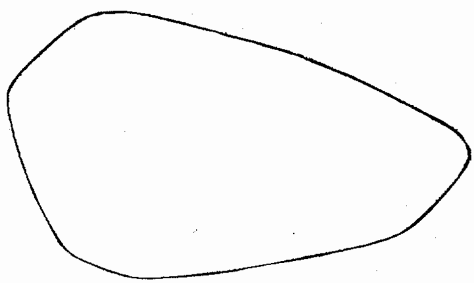

FIGURE 6

Kuiper [16] proved that each $\gamma \in \Gamma$ is semisimple and represented by a diagonal matrix over $\mathbf{R}$. In that case γ has three fixed points p_0, p_1, p_2, joined by invariant lines l_0, l_1, l_2 in $\mathbf{RP}^2$. The intersection of l_2 with Ω covers a closed geodesic curve $l_M(\gamma)$ on M. By grafting special $\mathbf{RP}^2$-annuli to M along $l_M(\gamma)$, we construct all $\mathbf{RP}^2$-manifolds with holonomy φ.

4.2. Let M be a convex $\mathbf{RP}^2$-manifold. Then by Kuiper [16], each simple closed curve α is represented by a unique closed geodesic $\alpha(M)$ which is simple. Furthermore the holonomy transformation can be represented by a diagonalizable matrix in GL(3; $\mathbf{R}$), and the intersection of Ω with the configuration of the three invariant lines is a line segment with endpoints the attracting and repelling fixed points (which covers the closed geodesic).

It is easy to see that $\alpha(M)$ has a neighborhood in M which extends to an $\mathbf{RP}^2$-annulus. In particular if w is a completely even positive word, then let $A_{\alpha(M)}(w)$ denote the unique such $\mathbf{RP}^2$-annulus which corresponds to w. Thus we can form the sum $M|\alpha(M) + A_{\alpha(M)}(w)$ which we shall denote by $M[\alpha]$ as before. Now consider a disjoint collection α_i, $i = 1, 2, 3, \cdots, n$ of homotopically nontrivial simple closed curves, no two of which are isotopic. For each $i = 1, 2, 3, \cdots, n$ choose a completely even positive word w_i. We shall call such a collection $\eta = (\alpha_1, \cdots, w_1, \cdots)$ a *grafting data* and we denote the set of all grafting data on M by $\mathscr{GD}(M)$. Let A_i denote the corresponding $\mathbf{RP}^2$-annulus corresponding to w_i which contains a closed geodesic with neighborhood isomorphic to that of $\alpha_i(M)$. Then we may simultaneously graft all the A_i to form a new $\mathbf{RP}^2$-manifold with holonomy φ. We denote this $\mathbf{RP}^2$-manifold by $M[\eta]$.

4.3. Theorem R. *Let M_0 be a compact convex $\mathbf{RP}^2$-manifold and let M be a compact $\mathbf{RP}^2$-manifold having the same holonomy representation. Then there exists a grafting data $\eta \in \mathscr{GD}(M)$ such that $M = M[\eta]$.*

4.4. The proof of Theorem R will be completely analogous to the proof of the Fuchsian surgery theorem for $\mathbf{CP}^1$-manifolds. After lifting to an S^2-bundle, we find an oriented 2-disk bundle (at least up to homotopy) and by calculating its Euler class in two ways deduce the conclusion of Theorem R. Before pursuing this analogy, we must explain how to use the fact that Γ preserves a convex domain to define a suitable Euler class.

If Ω is a strictly convex domain in $\mathbf{RP}^2$, which is invariant under a representation φ, then we decompose the flat $\mathbf{RP}^2$-bundle $\mathbf{RP}^2_\varphi$ over M into subbundles in the following way. The *convex decomposition* of $\mathbf{RP}^2$ is the decomposition $\mathbf{RP}^2 = \Omega \cup \Omega^c$. Since this decomposition is Γ-invariant, there is a corresponding decomposition of the flat $\mathbf{RP}^2$-bundle $\mathbf{RP}^2_\varphi$ over M as the union of the 2-disc bundle Ω_M and a Moebius-band bundle Ω^c_M. The developing section $\text{Dev}: M \to \mathbf{RP}^2_\varphi$ induces a decomposition of M into subsets $M_\Omega = \text{Dev}^{-1}(\Omega_M)$, $M_c = \text{Dev}^{-1}(\Omega^c_M)$. As for the complex case, the following facts are easily proven:

(1) Ω_M is a disjoint union of complete hyperbolic surfaces with ideal boundary a union of components of $\partial\Omega^c_M$;

(2) each component of Ω^c_M is an annulus;

(3) no component of Ω_M is simply connected.

4.5. Let $\widehat{\mathbf{RP}}^2$ denote the double cover of $\mathbf{RP}^2$. Since M is orientable, Γ preserves an orientation on Ω. The obstruction to lifting φ to an action on the double cover $\widehat{\mathbf{RP}}^2$ is exactly the second Stiefel-Whitney class $w_2(\varphi)$. Since a tubular neighborhood of the developing section of M_0 is isomorphic to the tangent microbundle of M, we can conclude that $w_2(\Omega_\varphi) = w_2(TM_0) = 0$. Thus Γ lifts to an action on $\widehat{\mathbf{RP}}^2$, and the developing section lifts to a section of the associated bundle $\widehat{\mathbf{RP}}^2_M$.

Since M is orientable, there are two components of the lift of Ω to $\mathbf{RP}^2$, which we call Ω_+, Ω_-. The lift of Ω^c is an annulus $\mathbf{A}$ which separates Ω_+ from Ω_-. As above we find a decomposition of M as $M = M_+ \cup M_{\mathbf{A}} \cup M_-$. Now all of the components of $M|\partial\Omega_M$ are oriented, and we obtain an orientation on M by requiring that the components of M_+ be positively oriented and those of M_- be negatively oriented. We have the following analogue of §2.7, whose proof is entirely analogous.

Proposition. *Let $e(\varphi)$ denote the Euler number of the oriented 2-disc bundle Ω_φ. Then $e(\varphi) = \chi(M_+) - \chi(M_-)$.*

Combining Proposition 4.5 with remarks (1), (2), (3) in 4.4, we conclude, just as in Theorem C, that either M_- or M_+ is a union of annuli. Suppose that it is M_+ which is a union of annuli. The $\mathbf{RP}^2$-structure of each component of M_+ is described by a completely even positive word; thus the structure on M_+ as well as a description of how M_+ is attached to M_- is encoded in a grafting data $\sigma \in \mathcal{GD}$. Then M is obtained from the convex structure M_0 by grafting along σ. Since the further details of the proof are parallel to those of the proof of Theorem C we omit them.

4.6. Construction of real projective structures. We have seen that given an $\mathbf{RP}^2$-manifold M with quadric holonomy, there is a canonical decomposition of M as a union of complete hyperbolic manifolds attached along their common ideal boundary. We shall further examine this decomposition. Let $\{M_i\}_{i\in I}$ be a collection of complete hyperbolic manifolds, such that each M_i admits a compactification M_i^- by ideal boundary components ∂M_i. Suppose we have partitioned the A_i into two disjoint classes labeled by sets I_+ and I_-. Let M^- denote the disjoint union of all M_i^-. Suppose given an involution $j\colon \partial M^- \to \partial M^-$ which has the property that if $i \in I_+$ (resp. $i \in I_-$), then each component of $j(\partial M_i)$ lies in ∂M_k where $k \in I_-$ (resp. where $k \in I_+$). For each j-invariant pair $(\partial\mu, \partial_{ij(\mu)})$ of boundary components, choose an annulus A_μ, and form the quotient space M of $M^- \cup \bigcup A_\mu$ by the identification determined by j. Clearly M is a smooth surface. Furthermore it is clear that by taking the surfaces M_i to be hyperbolic surfaces with the associated $\mathbf{RP}^2$-structures and the annuli A_μ to be suitable pieces of $\mathbf{RP}^2$-annuli, M can be

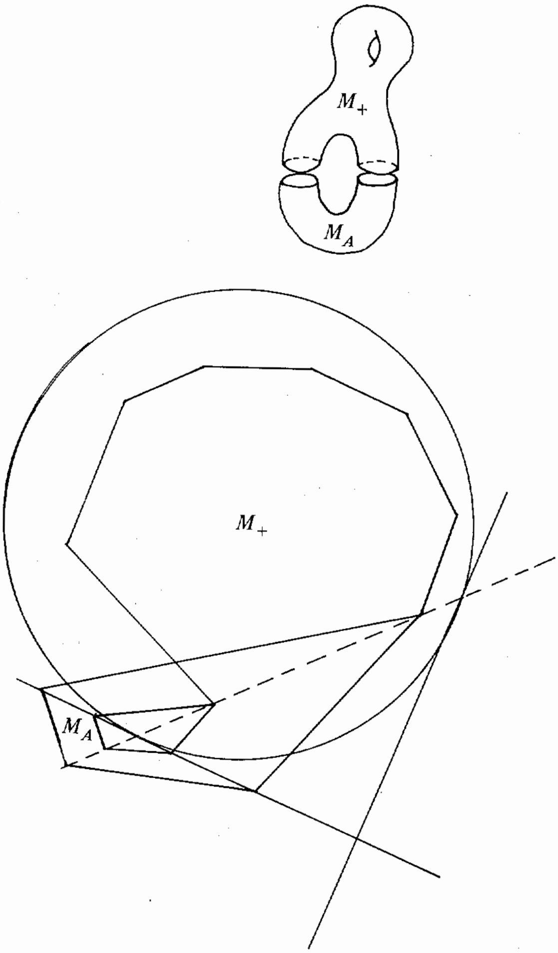

FIGURE 7

given an $\mathbf{RP}^2$-structure with quadric holonomy. Some examples are given in Figures 7–8. In particular Figure 8 indicates how to build an $\mathbf{RP}^2$-surface whose holonomy is a Schottky group in SO(1, 2).

We have seen how this decomposition determines the projective structure in the case of Fuchsian holonomy. Indeed, the decomposition of M as a union of compactified hyperbolic surfaces along ideal boundary components determines two types of data: (i) the pattern of annular components, described by a completely even positive word, (ii) the singular hyperbolic structure on M constructed as the union of the convex cores of the M_i. The singularities of this hyperbolic structure are all folds along boundary components. In §2 we saw

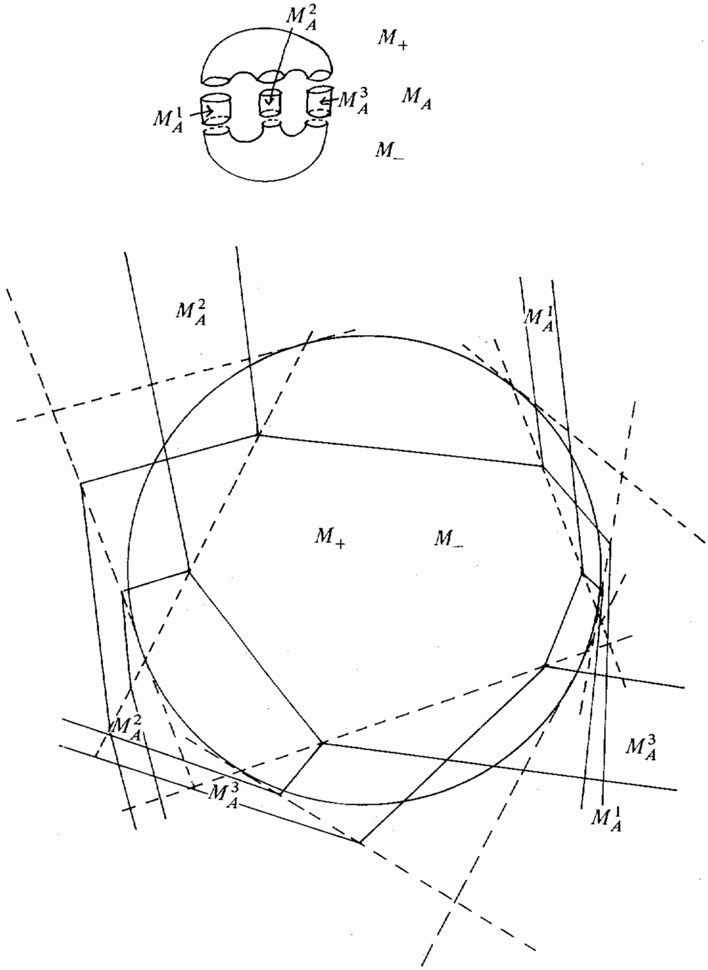

FIGURE 8

that there was a similar breakdown for a $\mathbf{CP}^1$-manifold with real holonomy. In particular, we have the following.

4.7. Theorem. *Let S be a closed orientable surface, $\chi(S) < 0$, and let $\pi = \pi_1(S)$. Let $\varphi \in \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbf{R}))$. Let $\mathscr{E}: \mathrm{PSL}(2, \mathbf{R}) \to \mathrm{SO}(2,1)^0$ be an isomorphism. Then the following are equivalent*:

(1) *$\mathscr{E} \circ \varphi$ is the holonomy of an $\mathbf{RP}^2$-structure on S*;

(2) *φ is the holonomy of a $\mathbf{CP}^1$-structure on S.*

In [5] it is shown that the following condition is also equivalent:

(3) $w_2(\phi) = 0$ (i.e., ϕ lifts to $\mathrm{SL}(2, \mathbf{R})$ and $\phi(\pi)$ is not a solvable group.

5. Examples of conformally flat manifolds

In this final section we shall show the same geodesic construction applies to geodesic structures in higher dimensions. Recall that a *flat conformal structure* on an n-dimensional manifold M is a geometric structure modelled on the n-sphere S^n, with coordinate changes lying in the group $\mathrm{Conf}(S^n)$ of conformal transformations of S^n. It is well known that $\mathrm{Conf}(S^n)$ is isomorphic to the orthogonal group $\mathrm{SO}(n+1,1)$. (A flat conformal structure can also be defined as a conformal class of Riemannian metrics, which are locally conformally Euclidean. Although this definition clarifies the Riemannian-geometry importance of these structures, we will not need these ideas here.) For more information on flat conformal structures, see Goldman [9], Johnson-Millson [12], Kulkarni [17], Kulkarni-Pinkall [18], Kamishima [14].

5.1. To begin, we consider the conformal geometry of the complement of a geometric k-sphere in S^n. Observe that the group G of conformal automorphisms of S^n which preserve S^k may be identified with the subgroup G of $\mathrm{O}(k+1,1) \times \mathrm{O}(n-k)$ with determinant 1. Furthermore, in the usual projective model where S^n is a quadric hypersurface in $\mathbf{RP}^{n+1}$, the intersections of S^n with a $(k+2)$-dimensional linear subspace containing S^k is a $(k+1)$-dimensional hemisphere. In particular the complement $S^n \setminus S^k$ is fibered by these $(k+1)$-discs, which are parametrized by S^{n-k-1}. In particular $S^n \setminus S^k \approx D^{k+1} \times S^{n-k-1}$ with the natural (product) action of $\mathrm{SO}(k+1,1) \times \mathrm{SO}(n-k)$. We may give $D^{k+1} \times S^{n-k-1}$ the product metric, where D^{k+1} has the Poincaré metric and S^{n-k-1} the spherical metric; then G acts isometrically on this product.

Now let $\Gamma \subset \mathrm{SO}(k+1,1)$ be the inclusion of a torsionfree discrete cocompact subgroup. Thus D^{k+1}/Γ is a compact hyperbolic manifold M^{k+1} and the quotient of $S^n \setminus S^k$ by $\Gamma \subset \mathrm{SO}(k+1,1) \subset \mathrm{SO}(n+1,1)$ is $M^{k+1} \times S^{n-k-1}$. Thus the product of a hyperbolic manifold with a sphere (or any spherical space form) is conformally flat.

5.2. Such product manifolds (when $n-k=2$) are the building blocks of our examples of conformally flat manifolds with surjective developing maps. In particular when $n>2$, these are the simplest examples of conformally flat manifolds whose developing maps are not injective maps from the universal cover to S^n.

Kulkarni [17] has defined a notion of connection sum of flat conformal structures. That is, if $B_1 \subset M_1$ and $B_2 \subset M_2$ are embeddings of geometric balls in two conformally flat manifolds M_1, M_2, then there exists a conformally flat structure on $M_1 \# M_2 = M_1 \setminus B_1 \cup M_2 \setminus B_2$, which restrict to the given flat conformal structures on each $M_i \setminus B_i$.

If $\text{dev}_i : \tilde{M}_i \to S^n$ $(i = 1, 2)$ are the developing maps of the M_i, and $\tilde{B}_i$ is a lift of $B_i \subset M_i$ to $\tilde{M}_i$, then there exists $h \in \text{Conf}(S^n)$ such that $h \circ \text{dev}(\tilde{B}_1) = \text{dev}(\tilde{B}_2)$. Let $j \in \text{Conf}(S^n)$ be inversion in $\partial\text{dev}(\tilde{D}_2)$. Then $j \circ h$ is a conformal map which defines a conformal isomorphism $\partial(\tilde{M}_1) \to \partial(\tilde{M}_2)$. The resulting identification space $M = M_1 \setminus B_1 \cup M_2 \setminus B_2$ thus inherits a flat conformal structure. If the developing maps of M_1 and M_2 are both injective, then so is the developing map of M. For a detailed proof, see Kulkarni-Pinkall [18].

Let M_1 be a conformally flat manifold homeomorphic to the product of S^1 with a closed hyperbolic $(n-1)$-manifold. Let M_1' be the k-fold covering space of M_1 induced from the k-fold covering map $S^1 \to S^1$. Let M_2 be any closed conformally flat manifold other than S^n. One can show that any conformally flat connected sum $M = M_1' \cup M_2$ must have surjective developing map.

To build an example on a manifold which is not a connected sum, we use a variant of the grafting construction. Let M_0 be a closed hyperbolic n-manifold and suppose that $F \subset M$ is a hypersurface satisfying the following condition:

(*) There exists a lift $\tilde{F}$ of F to the universal cover $\tilde{M}$ such that the intersection Λ of the closure of $\text{dev}(\tilde{F})$ with $S^{n-1} = \partial D^n$ is homeomorphic to S^{n-2}.

As above the conformally flat manifold $M' = (S^n \setminus \Lambda)/\pi_1(F) \approx F \times S^1$. Let $F' \subset M'$ be the hypersurface in M' which develops to $\text{dev}(\tilde{F}')$. The split manifold $M'|F'$ is a flat conformal manifold homeomorphic to $F \times I$, with two boundary components which develop to $\text{dev}(\tilde{F})$. Consider the split hyperbolic manifold $M_0 \setminus F$. Clearly there are collar neighborhoods of the respective boundary components of $M_0 \setminus F$ and $M' \setminus F'$ which are conformally equivalent. Thus the union $M[F] = M' \setminus F' \cup M_0 \setminus F$ has a conformally flat structure.

It is easy to see that the developing map of $M[F]$ is surjective (and hence not a covering map). For the developing image of M' is the complement of Λ in S^n. Thus the developing image of M contains the complement of Λ in S^n. However, the holonomy group of $M[F]$ is the same as holonomy group of M_0, i.e. Γ. Since the developing image of M is Γ-invariant, it must contain the union of all translates $\gamma(S^n \setminus \Lambda)$ $(\gamma \in \Gamma)$, which is easily seen to be all of S^n, since Γ acts minimally on S^{n-1}. Thus the developing map of M is surjective.

There are several important cases when (*) is satisfied. When $n = 3$ and F is an incompressible surface in M which is not the fiber of a fibration $M \to S^1$, then it follows from results of Maskit and Thurston that $\pi_1(F)$ is represented as a quasi-Fuchsian group acting on D^3, so (*) is satisfied. In another direction, if M happens to possess a totally geodesic hypersurface, we take F to be that hypersurface. Then Λ is actually a geometric sphere and (*) is satisfied in this case. Many of the "arithmetic" hyperbolic manifolds (i.e. hyperbolic manifolds of the form D^n/Γ, Γ an arithmetic lattice in $O(n,1)$) possess totally geodesic hypersurfaces; see Johnson-Millson [12] for a thorough discussion. In particular we obtain:

Theorem. *Let M be a closed hyperbolic manifold with holonomy representation $\varphi:\pi_2(M) \to Conf(S^{n-1})$. Suppose that either:*

(i) $\dim M = 3$ *and M contains an incompressible surface which is not the fiber of a fibration $M \to S^1$; or*

(ii) *M possesses a totally geodesic hypersurface.*

Then there exists a flat conformal structure on M with surjective developing map whose holonomy representation is the composition of φ with the inclusion $\mathrm{Conf}(S^{n-1}) \subset \mathrm{Conf}(S^n)$.

In case (ii) there is a whole 1-parameter family of flat conformal structures on M obtained by "bending" M along F, as discussed in Johnson-Millson [12] and Kourouniotis [15]. This is the analogue of "inserting θ-annuli" described in §2. In particular the structure we obtain above is obtained by "bending" M along F a full 2π radians. (The only angles through which one may bend and obtain the same holonomy representation are multiples of 2π.)

We have been unable as of yet to prove the following conjecture, which is the analogue of the Fuchsian surgery theorems for conformally flat structures on hyperbolic manifolds.

Conjecture. Let M_0 be a compact hyperbolic h-manifold and let $\varphi:\pi \to \mathrm{Conf}(S^{n-1})$ be its holonomy representation. Suppose that M is a conformally flat manifold whose holonomy is the composition of φ with the inclusion $\mathrm{Conf}(S^{n-1} \subset \mathrm{Conf}(S^n)$. Then there exists a (not necessarily connected) hypersurface $F \subset M_0$ such that M is obtained from M_0 by grafting along F.

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UNIVERSITY OF MARYLAND